Answers to Coursebook questions – Chapter H4

1 **a** $E = m_0c^2 = 9.1\times10^{-31}\times9\times10^{16} = 8.19\times10^{-14}\ \text{J} = 5.1\times10^{5}\ \text{eV} = 0.51\ \text{MeV}$.

b **i** Using classical mechanics: the work done is the change in kinetic energy and so

$$\frac{1}{2}mv^2 = qV \Rightarrow v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2\times1.6\times10^{-19}\times1.00\times10^{9}}{9.1\times10^{-31}}} = 1.9\times10^{10}\ \text{m s}^{-1}$$

which is greater than the speed of light.

ii Using relativity: the work done is $1.00\times10^{9}\ \text{eV} = 1.00\times10^{3}\ \text{MeV}$ and this is the change in kinetic energy, i.e.

$$E_K = (\gamma-1)\times0.511 = 1.00\times10^{3} \Rightarrow \gamma-1 = \frac{1000}{0.511} \Rightarrow \gamma = 1958.$$

This implies that the speed of the electron is essentially the speed of light: $1958 = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \Rightarrow 1-\dfrac{v^2}{c^2} = \dfrac{1}{1958^2} \Rightarrow \dfrac{v}{c} = \sqrt{1-\dfrac{1}{1958^2}} = 0.99999987$.

2 $E_K = (\gamma-1)m_0c^2 = 10m_0c^2 \Rightarrow \gamma = 11$.

Hence, $11 = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \Rightarrow 1-\dfrac{v^2}{c^2} = \dfrac{1}{11^2} \Rightarrow \dfrac{v}{c} = \sqrt{1-\dfrac{1}{11^2}} = 0.996$.

3 $E_T = 5m_0c^2 = 5\times938 = 4690\ \text{MeV}$.

From $E^2 = (m_0c^2)^2 + p^2c^2$, $pc = \sqrt{E^2-(m_0c^2)^2} = \sqrt{4690^2-938^2} = 4595\ \text{MeV}$.

Hence, $p = 4.6\times10^{3}\ \text{MeV c}^{-1}$.

4 The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.9^2}} = 2.29$.

Hence, $E_T = \gamma m_0c^2 = 2.29\times0.511 = 1.17 \approx 1.2\ \text{MeV}$.

5 $p = 685\ \text{MeV c}^{-1} = \dfrac{685\times10^{6}\times1.6\times10^{-19}}{3.0\times10^{8}} = 3.7\times10^{-19}\ \text{N s}$.

6 The total energy is $E = \sqrt{(m_0c^2)^2 + p^2c^2} = \sqrt{938^2+500^2} = 1063\ \text{MeV}$.

Hence, $E_K = 1063 - 938 = 125\ \text{MeV}$.

7

This question is a careless remnant of the old syllabus in which the mass was assumed to change with speed and $m = \gamma m_0$. In other words, $E_T = mc^2 = \gamma m_0 c^2$.

$$m = \frac{E_T}{c^2} = \frac{10.0\times10^9 \times 1.6\times10^{-19}}{9\times10^{16}} = 1.8\times10^{-26}\ \text{kg}.$$

8 Both are essentially the speed of light, since the total energy is so much greater than the rest energy. A calculator would give $v = c$, a computer gives the answers in the textbook (see page 815 in *Physics for the IB Diploma*).

9 The work done is $2.0\ \text{MeV}$ and this is the change in kinetic energy. Since the electron was accelerated from rest this is also the final kinetic energy.

Hence $E_K = (\gamma - 1)m_0c^2 = (\gamma - 1)\times 0.511\ \text{MeV} = 2.0\ \text{MeV}$,

leading to $\gamma - 1 = 3.91 \Rightarrow \gamma = 4.91$.

Therefore, $4.91 = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \Rightarrow 1 - \dfrac{v^2}{c^2} = \dfrac{1}{4.91^2} \Rightarrow \dfrac{v^2}{c^2} = 1 - \dfrac{1}{4.91^2} = 0.9585$

and so $\dfrac{v}{c} = \sqrt{0.9585} = 0.979 \approx 0.98$.

10 The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.96^2}} = 3.57$.

Hence the kinetic energy is $E_K = (\gamma - 1)m_0c^2 = 2.57\times 938 = 2.4\times10^3\ \text{MeV}$. The change in kinetic energy is the work done which in turn equals qV. Hence $V = 2.4\times10^3\ \text{MV}$.

11 The gamma factor is $\gamma = \dfrac{1}{\sqrt{1-0.99^2}} = 7.1$.

Hence the kinetic energy is $E_K = (\gamma - 1)m_0c^2 = 6.1\times 938 = 5.7\times10^3\ \text{MeV}$.

12 The gamma factors in each case are

a $\gamma = \dfrac{1}{\sqrt{1-0.50^2}} = 1.155$

b $\gamma = \dfrac{1}{\sqrt{1-0.90^2}} = 2.294$

c $\gamma = \dfrac{1}{\sqrt{1-0.99^2}} = 7.089$

So in each case the kinetic energy of the electron would be

a $E_K = (\gamma - 1)m_0c^2 = 0.155 \times 0.511 = 0.079\ \text{MeV}$ requiring an accelerating voltage of $0.079\ \text{MV}$.

b $E_K = (\gamma - 1)m_0c^2 = 1.294 \times 0.511 = 0.661\ \text{MeV}$ requiring an accelerating voltage of $0.661\ \text{MV}$.

c $E_K = (\gamma - 1)m_0c^2 = 6.089 \times 0.511 = 3.11\ \text{MeV}$ requiring an accelerating voltage of $3.11\ \text{MV}$.

13 **a** The work done is

$W = Fd = eEd = 1.6 \times 10^{-19} \times 5.0 \times 10^6 \times 10^3 = 8.0 \times 10^{-10}\ \text{J} = 5.0 \times 10^9\ \text{eV} = 5.0 \times 10^3\ \text{MeV}$

Therefore $E_K = 5.0 \times 10^3\ \text{MeV}$ (or 5.0 GeV).

b The total energy is $E_T = (938 + 5.0 \times 10^3) = 5.938 \times 10^3\ \text{MeV}$.

Hence $\gamma = \dfrac{E_T}{m_0c^2} = \dfrac{5.938 \times 10^3}{938} = 6.33$.

Hence the speed is $6.33 = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} \Rightarrow 1 - \dfrac{v^2}{c^2} = \dfrac{1}{6.33^2} \Rightarrow \dfrac{v}{c} = \sqrt{1 - \dfrac{1}{6.33^2}} = 0.987$.

14 **a** From $p = \gamma m_0 v$ and $E = \gamma m_0 c^2$ we get by dividing side by side to get rid of the gamma factor: $\dfrac{p}{E} = \dfrac{v}{c^2}$. Hence, $v = \dfrac{pc^2}{E} = \dfrac{pc^2}{\sqrt{m_0^2c^4 + p^2c^2}}$ as required.

b For the electron with momentum $1.00\ \text{MeV c}^{-1}$,

$\sqrt{m_0^2c^4 + p^2c^2} = \sqrt{0.511^2 + 1.00^2} = 1.123$.

For the proton, $\sqrt{m_0^2c^4 + p^2c^2} = \sqrt{938^2 + 1.00^2} \approx 938$.

Hence the ratio of the speeds is $\dfrac{v_e}{v_p} = \dfrac{938}{1.123} = 835$.

c For a momentum of $1.00\ \text{GeV c}^{-1}$, for the electron

$\sqrt{m_0^2c^4 + p^2c^2} = \sqrt{0.511^2 + (10^3)^2} \approx 10^3$ and for the proton

$\sqrt{m_0^2c^4 + p^2c^2} = \sqrt{938^2 + (10^3)^2} = 1371$ so that $\dfrac{v_e}{v_p} = \dfrac{1371}{1000} = 1.37$.

d As the momentum increases we may neglect the rest energy in which case both speeds tend to become the speed of light and so the ratio approaches 1.

15 a The momentum of the fragments must be zero since the original momentum before the breakup was zero. The gamma factor is

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.85^2}} = 1.90$$

and so the momentum is $1.90 \times 125\ \text{MeV c}^{-2} \times 0.85c = 202\ \text{MeV c}^{-1}$. This is also the momentum of the other fragment. Hence the total energy of each of the fragments is $E_1 = \sqrt{125^2 + 202^2} = 238\ \text{MeV}$ and $E_2 = \sqrt{250^2 + 202^2} = 321\ \text{MeV}$.

b The total energy of the system therefore is $238 + 321 = 559\ \text{MeV}$. This is the rest energy of the particle that broke up and so its rest mass is $559\ \text{MeV c}^{-2}$.

16 a The total momentum of the electron–positron pair is zero. If only one photon is produced it will have momentum violating the law of momentum conservation.

b Again because of momentum conservation.

c The total energy of the electron is $E = mc^2 + E_K = 0.51 + 2.0 = 2.51\ \text{MeV}$.
The positron has the same energy.
The total energy is then $E_T = 2 \times 2.51 \approx 5.0\ \text{MeV}$.
The photons must have the same energy because they move in opposite directions with the same momentum (magnitude) and hence the same wavelength. So each has an energy of about $2.5\ \text{MeV}$.

17 a Initially the electron has zero momentum.
The photon has energy $\frac{hc}{\lambda}$ and momentum $\frac{h}{\lambda}$ in the *x*-direction.
After the interaction, the electron moves with some kinetic energy and the photon has energy $\frac{hc}{\lambda'}$ and momentum $\frac{h}{\lambda'}$ in the direction shown in the diagram (see page 676 in *Physics for the IB Diploma*).
Conservation of momentum implies that

$$\frac{h}{\lambda} = \frac{h}{\lambda'}\cos\theta + p_x$$

$$0 = \frac{h}{\lambda'}\sin\theta - p_y$$

From these two equations we deduce that

$$\begin{aligned} p^2 &= p_x^2 + p_y^2 \\ &= \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\cos\theta\right)^2 + \left(\frac{h}{\lambda'}\sin\theta\right)^2 \\ &= \left(\frac{h}{\lambda}\right)^2 - \frac{2h^2}{\lambda\lambda'}\cos\theta + \left(\frac{h}{\lambda'}\right)^2 \end{aligned}$$

Conservation of energy implies $\frac{hc}{\lambda}+m_0c^2=\frac{hc}{\lambda'}+\sqrt{(m_0c^2)^2+p^2c^2}$,

and so $(m_0c^2)^2+p^2c^2=(\frac{hc}{\lambda}-\frac{hc}{\lambda'}+m_0c^2)^2$.

Hence, using the expression for momentum above,

$(m_0c^2)^2+(\frac{hc}{\lambda})^2-\frac{2h^2c^2}{\lambda\lambda'}\cos\theta+(\frac{hc}{\lambda'})^2=(\frac{hc}{\lambda}-\frac{hc}{\lambda'}+m_0c^2)^2$. Expanding all,

$$(m_0c^2)^2+(\frac{hc}{\lambda})^2-\frac{2h^2c^2}{\lambda\lambda'}\cos\theta+(\frac{hc}{\lambda'})^2=(\frac{hc}{\lambda})^2+(\frac{hc}{\lambda'})^2+(m_0c^2)^2-\frac{2h^2c^2}{\lambda\lambda'}+2\frac{hc}{\lambda}m_0c^2-2\frac{hc}{\lambda'}m_0c^2$$

$$-\frac{2h^2c^2}{\lambda\lambda'}\cos\theta=-\frac{2h^2c^2}{\lambda\lambda'}+2\frac{hc}{\lambda}m_0c^2-2\frac{hc}{\lambda'}m_0c^2$$

$$\lambda'-\lambda=\frac{h}{m_0c}(1-\cos\theta)$$

b Some of the energy of the incoming photon has been used to provide the kinetic energy of the electron and so the scattered photon has reduced energy.

Since the energy of a photon is $E=\frac{hc}{\lambda}$ the wavelength has increased.

c Applying the formula just derived,

$$\lambda'-\lambda=\frac{h}{m_0c}(1-\cos\theta)$$

$$=\frac{6.63\times10^{-34}}{9.1\times10^{-31}\times3.0\times10^{8}}\times\left(1-\frac{1}{2}\right)$$

$$=1.21\times10^{-12}\text{ m}$$

Hence $\lambda'=3.00\times10^{-12}+1.21\times10^{-12}=4.21\times10^{-12}$ m.

18 The initial momentum of the photon is

$$p = \frac{h}{\lambda}$$
$$= \frac{6.63\times10^{-34}}{5.0\times10^{-12}}$$
$$= 1.326\times10^{-22}\ \text{N s}$$

Because

$$\lambda' - \lambda = \frac{h}{m_0 c}(1-\cos\theta)$$
$$= \frac{6.63\times10^{-34}}{9.1\times10^{-31}\times3.0\times10^{8}}\times(1-(-1))$$
$$= 4.857\times10^{-12}\ \text{m}$$

The wavelength of the scattered photon is therefore $\lambda' = 9.857\times10^{-12}$ m.
The momentum of the scattered photon has magnitude

$$p = \frac{h}{\lambda}$$
$$= \frac{6.63\times10^{-34}}{9.857\times10^{-12}}$$
$$= 0.673\times10^{-22}\ \text{N s}$$

And so the change in the momentum of the photon is

$$\Delta p = 1.326\times10^{-22} - (-0.673\times10^{-22})$$
$$= 2.0\times10^{-22}\ \text{N s}$$

This means that the electron moves off with this momentum. The total energy of the electron is therefore
$E_T = \sqrt{(9.1\times10^{-31}\times9\times10^{16})^2 + (2.0\times10^{-22}\times3\times10^{8})^2} = 1.015\times10^{-13}\ \text{J} = 0.6345\ \text{MeV}$.
Hence the kinetic energy is $E_K = 0.6345 - 0.511 = 0.124$ MeV.

The gamma factor is $\gamma = \frac{0.6345}{0.511} = 1.242$.

Hence, $1.242 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{1.242^2} \Rightarrow \frac{v}{c} = \sqrt{1-\frac{1}{1.242^2}} = 0.593$.

19 The gamma factor is $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.80^2}} = \frac{5}{3}$

and so the total energy of the pion is $E_T = \frac{5}{3} \times 135 = 225\text{ MeV}$. The momentum of the pion is: $225 = \sqrt{(135)^2 + (pc)^2} \Rightarrow pc = \sqrt{225^2 - 135^2} = 180\text{ MeV} \Rightarrow p = 180\text{ MeV c}^{-1}$.
Conservation of energy and momentum gives:

$$225 = hf_A + hf_B$$

$$180 = \frac{hf_A}{c} - \frac{hf_B}{c}$$

To simplify things set c = 1, which is all right since we are going to take a ratio and units will not be important. Then

$$225 = hf_A + hf_B$$

$$180 = hf_A - hf_B$$

Adding, $f_A = \frac{405}{2h}$, subtracting, $f_B = \frac{45}{2h}$. The ratio is therefore $\frac{f_A}{f_B} = \frac{405}{45} = 9$.

20 The momentum of the two bodies is zero and so the particle they form is produced at rest. The gamma factor is $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.80^2}} = \frac{5}{3}$ and so the momentum of each particle is $\frac{5}{3} \times 3.0 \times 0.80 \times 3 \times 10^8 = 1.2 \times 10^9\text{ N s}$.
Hence the total energy of each of the particles is
$E = \sqrt{(3.0 \times 9.0 \times 10^{16})^2 + (1.2 \times 10^9 \times 3 \times 10^8)^2} = 4.5 \times 10^{17}\text{ J}$.
The rest energy of the particle that is formed and so its rest mass is therefore
$2 \times 4.5 \times 10^{17} = 9.0 \times 10^{17}\text{ J}$ and hence the rest mass is $\frac{9.0 \times 10^{17}}{9.0 \times 10^{16}} = 10\text{ kg}$.

21 **a** $p = \gamma m_0 v$

b $E = \gamma m_0 c^2$

c Eliminating the gamma factor from **a** and **b** we get $\frac{p}{E} = \frac{v}{c^2}$.

Hence $v = \frac{pc^2}{E}$ as required.

d For a massless particle, $E = \sqrt{0 + p^2c^2} = pc$, hence $v = \frac{pc^2}{E} = \frac{pc^2}{pc} = c$.