Answers to Coursebook questions – Chapter 1.5

1 Let a be the side of the square and r the radius of the circle. Then $4a = 2\pi r \Rightarrow a = \frac{\pi r}{2}$.

The ratio of circle to square area is $\frac{\pi r^2}{a^2} = \frac{\pi r^2}{(\frac{\pi r}{2})^2} = \frac{4}{\pi} > 1$, so the circle has the greater area.

2 Let a be the side of the cube and r the radius of the sphere. Then,

$6a^2 = 4\pi r^2 \Rightarrow a = r\sqrt{\frac{2\pi}{3}}$.

The ratio of sphere to cube volume is $\frac{\frac{4\pi r^3}{3}}{a^3} = \frac{\frac{4\pi r^3}{3}}{r^3(\frac{2\pi}{3})^{3/2}} = \sqrt{\frac{6}{\pi}} > 1$,

so the sphere has the greater volume.

3 We have $\cos x \approx 1 - \frac{x^2}{2}$ and so $1 - \cos x \approx \frac{x^2}{2}$.

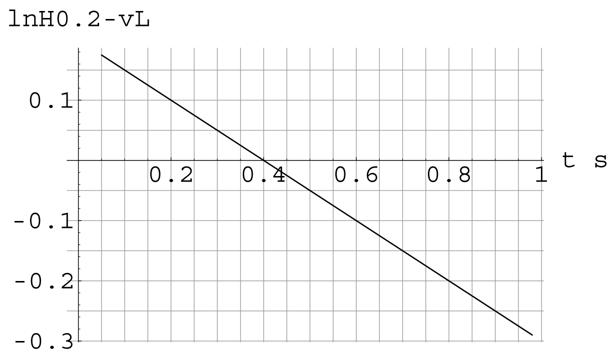
4 **a** The initial voltage V_0 is such that $\ln V_0 = 4 \Rightarrow V_0 = e^4 = 55\ \text{V}$.

b When $V = \frac{V_0}{2} \approx 27\ \text{V}$, $\ln V = \ln 27 \approx 3.29$. From the graph, when $\ln V \approx 3.29$ we find $t \approx 7\ \text{s}$.

c Since $V = V_0 e^{-t/RC}$, taking logs, $\ln V = \ln V_0 - \frac{t}{RC}$, so a graph of $\ln V$ versus time gives a straight line with slope equal to $-\frac{1}{RC}$. The slope of the given graph is approximately $\frac{4-2}{0-20} = -0.10$.

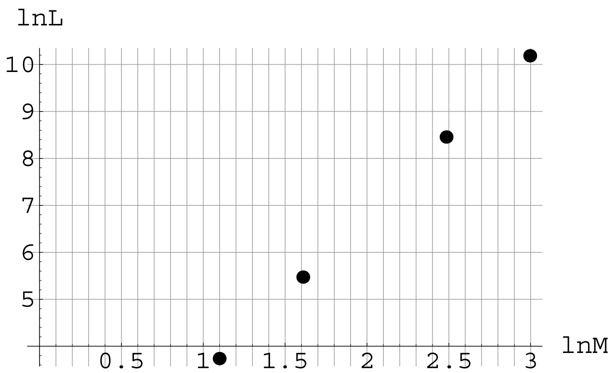
Hence, $-\frac{1}{RC} = -0.10 \Rightarrow R = \frac{1}{0.10 \times C} = \frac{1}{0.10 \times 5 \times 10^{-6}} = 2 \times 10^6\ \Omega$.

5 We expect $v = v_T(1 - e^{-at})$. The velocity approaches the terminal value $v_T = 0.2 \text{ m s}^{-1}$ and so $v = 0.2(1 - e^{-at})$. Hence, $v - 0.2 = -0.2e^{-at}$ and so $\ln(0.2 - v) = \ln 0.2 - at$. Graphing $\ln(0.2 - v)$ versus t gives a straight line with slope $-a$.

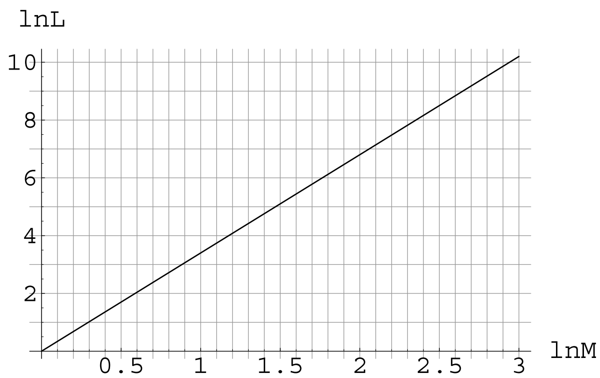


The slope of this line is $\dfrac{0.15 - (-0.25)}{0.1 - 0.9} = -0.5$; hence $a = 0.5 \text{ s}^{-1}$, so the equation is $v = 0.2(1 - e^{-0.5t})$.

6 We expect $L = kM^{\alpha}$, and so $\ln L = \ln k + \alpha \ln M$. A graph of $\ln L$ versus $\ln M$ is shown below. The slope is α .

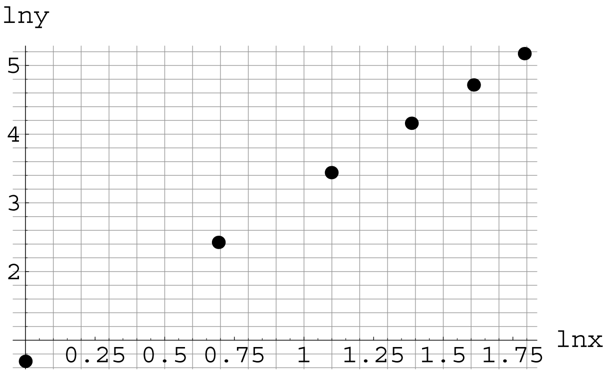


Drawing a line of best fit gives:



Measuring the slope gives $\alpha = 3.4$.

7 We can plot either y versus $x^{2.5}$ to get a straight line with slope c or $\ln y$ versus $\ln x$ with slope 2.5 and vertical intercept $\ln c$. Doing the latter gives:



The vertical intercept is about 0.70, and so $\ln c = 0.70 \Rightarrow c = e^{0.70} \approx 2.0$

8 We can plot either y versus e^{x^2} or $\ln y$ versus x^2.

9 $S = (120 + 60) \pm (5 + 3) = 180 \pm 8$. $D = (120 - 60) \pm (5 + 3) = 60 \pm 8$.

10 **a** $Q = Q_0 \pm \Delta Q$. $Q_0 = \frac{20}{10} = 2.0$. $\frac{\Delta a}{a_0} = \frac{1}{20} = 0.05$ and $\frac{\Delta b}{b_0} = \frac{1}{10} = 0.10$.

Hence, $\frac{\Delta Q}{Q_0} = \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0} = 0.05 + 0.10 = 0.15$. Hence,

$\Delta Q = Q_0 \times 0.15 = 2.0 \times 0.15 = 0.30$. Hence, $Q = 2.0 \pm 0.3$. (The uncertainty is expressed in 1 significant figure.)

b $Q = Q_0 \pm \Delta Q$. $Q_0 = 2 \times 20 + 3 \times 15 = 85$.

$\Delta Q = 2 \times \Delta a + 3 \times \Delta b = 2 \times 2 + 3 \times 3 = 13 \approx 10$. Hence, $Q = 85 \pm 10 \approx 80 \pm 10$. (The uncertainty is expressed in 1 significant figure. The uncertainty is large and affects the hundreds digit of the answer. Hence there is no point in quoting the answer in the units digit, hence the rounding.)

c $Q = Q_0 \pm \Delta Q$. $Q_0 = 50 - 2 \times 24 = 2$. $\Delta Q = \Delta a - 2 \times \Delta b = 1 + 3 \times 0.5 = 2.5 \approx 2$.

Hence, $Q = 2 \pm 2$. (The answer has no decimal places, so the uncertainty must be kept to the units digit.)

d $Q = Q_0 \pm \Delta Q$. $Q_0 = 10^2 = 100$. $\frac{\Delta a}{a_0} = \frac{0.3}{10.0} = 0.03$.

Hence, $\frac{\Delta Q}{Q_0} = 2 \times \frac{\Delta a}{a_0} = 2 \times 0.03 = 0.06$. Hence, $\Delta Q = Q_0 \times 0.06 = 100 \times 0.06 = 6$.

Hence, $Q = 100 \pm 6$.

e $Q = Q_0 \pm \Delta Q$. $Q_0 = \frac{100^2}{20^2} = 25$. $\frac{\Delta a}{a_0} = \frac{5}{100} = 0.05$ and $\frac{\Delta b}{b_0} = \frac{2}{20} = 0.10$.

Hence, $\frac{\Delta Q}{Q_0} = 2 \times \frac{\Delta a}{a_0} + 2 \times \frac{\Delta b}{b_0} = 0.10 + 0.20 = 0.30$.

Hence, $\Delta Q = Q_0 \times 0.30 = 25 \times 0.30 = 7.5 \approx 8$. Hence, $Q = 25 \pm 8$.

11 $F_0 = \frac{2.8 \times 14^2}{8.0} = 68.6\ \text{N}$. $F = F_0 \pm \Delta F$.

$$\frac{\Delta F}{F_0} = \frac{\Delta m}{m_0} + 2 \times \frac{\Delta v}{v_0} + \frac{\Delta r}{r_0} = \frac{0.1}{2.8} + 2 \times \frac{2}{14} + \frac{0.2}{8.0} = 0.346.$$

Hence, $\Delta F = F_0 \times 0.346 = 68.6 \times 0.346 = 23.7 \approx 20$ to one significant figure.

Hence, $F = (68.6 \pm 20)\ \text{N}$. Since the uncertainty is in the tens digit there is no point in quoting the answer beyond the tens digit. Hence, $F = (70 \pm 20)\ \text{N}$. Notice that, whereas each of mass, speed and radius are given to 2 s.f., the uncertainty in the force does not warrant keeping 2 s.f. in the answer.

12 The volume of the cube is
$V_0 = (60\times10^{-3})\times(50\times10^{-3})\times(40\times10^{-3}) = 1.2\times10^{-4}\ \text{m}^3$.

Hence, the density is $\rho_0 = \dfrac{m_0}{V_0} = \dfrac{2.2}{1.2\times10^{-4}} = 1.8\times10^4\ \text{kg m}^{-3}$.

The uncertainty in the density satisfies
$$\frac{\Delta\rho}{\rho_0} = \frac{\Delta m}{m_0} + \frac{\Delta a}{a_0} + \frac{\Delta b}{b_0} + \frac{\Delta c}{c_0} = \frac{0.2}{2.2} + \frac{3}{60} + \frac{2}{50} + \frac{2}{40} = 0.231.$$

Hence,
$\Delta\rho = \rho_0 \times 0.231 = 1.8\times10^4 \times 0.231 = 4.16\times10^3 \approx 0.4\times10^4$.

Hence, $\rho_0 = (1.8\pm0.4)\times10^4\ \text{kg m}^{-3}$.

13 **a** The area is $A_0 = \pi r_0^2 = \pi\times2.4^2 = 18.1\ \text{cm}^2$.

The uncertainty is $\dfrac{\Delta A}{A_0} = 2\times\dfrac{\Delta r}{r_0} = 2\times\dfrac{0.1}{2.4} = 0.083$, i.e.

$\Delta A = A_0 \times 0.083 = 18.1\times0.083 = 1.502$.

The answer must have at most 2 s.f., so $A = (18\pm2)\ \text{cm}^2$.

b The circumference is $C_0 = 2\pi r_0 = 2\pi\times2.4 = 15.1\ \text{cm}$, and
$\Delta C = 2\pi\Delta r = 2\pi\times0.1 = 0.628\ \text{cm}$. Hence, $C = (15\pm1)\ \text{cm}$.

14 **a** $A = 4\pi r^2$, and so $A = 4\pi\times22.7^2 = 6475.3\ \text{cm}^2$.

We have $\dfrac{\Delta A}{A} = 2\dfrac{\Delta r}{r}$, i.e. $\dfrac{\Delta A}{A} = 2\times\dfrac{0.2}{22.7} = 0.01762 \Rightarrow \Delta A = 114\ \text{cm}^2$.

We must express the uncertainty to 1 s.f., i.e. $\Delta A = 100 = 1\times10^2\ \text{cm}^2$.

This means that $A = (65\pm1)\times10^2\ \text{cm}^2$ or $A = (6.5\pm0.1)\times10^3\ \text{cm}^2$.

b Similarly, $V = \dfrac{4\pi r^3}{3} = \dfrac{4\pi\times22.7^3}{3} = 48997\ \text{cm}^3$ and $\dfrac{\Delta V}{V} = 3\dfrac{\Delta r}{r}$ so that

$\dfrac{\Delta V}{V} = 3\times\dfrac{0.2}{22.7} = 0.02643 \Rightarrow \Delta V = 1295\ \text{cm}^3$.

We must express the uncertainty to 1 s.f., i.e. $\Delta V = 1000 = 1\times10^3\ \text{cm}^3$. This means that $V = (49\pm1)\times10^3\ \text{cm}^3$ or $V = (4.9\pm0.1)\times10^4\ \text{cm}^3$.

15 The area is

$$A = 4.4 \times 8.5 = 37.4 \text{ cm}^2$$

$$\frac{\Delta A}{A} = \frac{\Delta a}{a} + \frac{\Delta b}{b} = \frac{0.2}{4.4} + \frac{0.3}{8.5} = 0.0807 \Rightarrow \Delta A = 3.02 \text{ cm}^2$$

We must express the uncertainty to 1 s.f., i.e.

$$\Delta A = 3 \text{ cm}^2$$

This means that

$$A = 37 \pm 3 \text{ cm}^2$$

b The perimeter is $P = 2a + 2b = 2 \times 4.4 + 2 \times 8.5 = 25.8 \text{ cm}$.
$\Delta P = 2 \times \Delta a + 2 \times \Delta b = 2 \times 0.2 + 2 \times 0.3 = 1.0 \text{ cm}$.
This means that $P = 26 \pm 1 \text{ cm}$.

16 By a factor of $\sqrt{2}$.

17 $\frac{\Delta T}{T} = \frac{1}{2}\frac{\Delta L}{L}$ and so $\frac{\Delta T}{T} = \frac{1}{2} \times 2\% = 1\%$.

18 $T = 2\pi\sqrt{\frac{L}{g}} \Rightarrow g = \frac{4\pi^2 L}{T^2}$. This implies that

$$\frac{\Delta g}{g} = \frac{\Delta L}{L} + 2\frac{\Delta T}{T} = 0.5 + 2 \times 0.6 = 1.7\% \approx 2\%.$$

