Self-assessment answers: 25 Mathematical induction

1. Proposition: $\sum_{r=1}^{n} r(r+2) = \frac{n(n+1)(2n+7)}{6}$

For $n = 1$:

LHS = $1 \times 3 = 3$; RHS = $\frac{1\times 2\times 9}{6} = 3$

Assume the proposition is true for $n = k$,

$$\sum_{r=1}^{k} r(r+2) = \frac{k(k+1)(2k+7)}{6}$$

Let $n = k + 1$. Working towards:

$$\sum_{r=1}^{k+1} r(r+2) = \frac{(k+1)(k+2)(2k+9)}{6}$$

LHS = $\sum_{r=1}^{k+1} r(r+2) = (k+1)(k+3) + \frac{k(k+1)(2k+7)}{6}$ using the assumption.

$$= \frac{k+1}{6}\left[6(k+3) + k(2k+7)\right]$$

$$= \frac{k+1}{6}\left[2k^2 + 13k + 18\right]$$

$$= \frac{(k+1)(k+2)(2k+9)}{6} = \text{RHS}$$

So the result is true for $n = 1$ and if true for $n = k$ then it is true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all $n \in \mathbb{Z}^+$.

[6 marks]

2. Proposition: $f(n) = 15^n - 2^n$ is divisible by 13.

$f(0) = 1 - 1 = 0 = 0 \times 13$, so $f(0)$ is divisible by 13.

Assume that $f(k)$ is divisible by 13. Then $f(k) = 13A$ for some $A \in \mathbb{Z}$.

When $n = k + 1$, we are working towards $f(k + 1) = 15^{k+1} - 2^{k+1} = 13B$ for some $B \in \mathbb{Z}$.

LHS $= 15 \times (15^k - 2^k) + 13 \times 2^{k+1}$ using the assumption.

$= 15 \times 13A + 13 \times 2^{k+1}$

$= 13(15A + 2^{k+1})$ = RHS with $B = 15A + 2^{k+1}$

So $f(k + 1)$ is divisible by 13.

$f(0)$ is divisible by 13 and if $f(k)$ is divisible by 13 then so is $f(k + 1)$.Therefore, by the principle of mathematical induction, $f(n)$ is divisible by 13 for all $n \in \mathbb{N}$. *[6 marks]*

3. Proposition: $f(n) = 3^n > n + 17$ for $n \geq 3$.

$f(3) = 27 > 20 = 3 + 17$, so the proposition is true for $n = 3$.

Assume that $f(k) > k + 17$.

When $n = k + 1$, we are working towards $f(k + 1) > k + 18$.

LHS $= 3 \times f(k) > 3 \times (k + 17)$, using the assumption

LHS $> k + 18 + (2k + 33) > k + 18$ since $2k + 33 > 0$ for $k \geq 3$.

LHS > RHS, so the proposition holds for $n = k + 1$.

The inequality holds for $n = 3$ and if it holds for $n = k$ then it also holds for $n = k + 1$. Therefore, by the principle of mathematical induction, it holds for all integers $n \geq 3$. *[6 marks]*

4. (a) Proposition: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

For $n = 0$:

LHS = 1; RHS = $\cos(0) + i\sin(0) = 1 + 0i = 1$.

Assume the proposition is true for $n = k$: $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$.

Let $n = k + 1$.

Working towards: $(\cos\theta + i\sin\theta)^{k+1} = \cos((k+1)\theta) + i\sin((k+1)\theta)$.

LHS = $(\cos\theta + i\sin\theta)[\cos(k\theta) + i\sin(k\theta)]$

$= \cos\theta\cos(k\theta) - \sin\theta\sin(k\theta) + i(\cos\theta\sin(k\theta) + \sin\theta\cos(k\theta))$

$= \cos((k+1)\theta) + i\sin((k+1)\theta)$ = RHS, by compound angle formulae.

So the result is true for $n = 0$ and if true for $n = k$ then it is true for $n = k + 1$. Therefore, by the principle of mathematical induction, it is true for all $n \in \mathbb{N}$.

(b) Using (a), $(\cos\theta + i\sin\theta)^5 = \cos(5\theta) + i\sin(5\theta)$.

Hence, $\sin(5\theta) = \text{Im}[(\cos\theta + i\sin\theta)^5]$

$= \text{Im}[\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta]$

$= 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1 - \sin^2\theta)^2\sin\theta - 10(1 - \sin^2\theta)\sin^3\theta + \sin^5\theta$

$5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$ *[12 marks]*

