Self-assessment answers: 15 Complex numbers

1. With $z = x + \mathrm{i}y$, $z^* = x - \mathrm{i}y$

$\Rightarrow\ z + 2z^* = 3x - \mathrm{i}y = -1 + 4\mathrm{i}$

$\Rightarrow\ x = -\frac{1}{3}, y = -4$

$\Rightarrow\ z = -\frac{1}{3} - 4\mathrm{i}$ *[5 marks]*

2. $(\text{cis}\ \theta)^n = \text{cis}\ n\theta$ and $\frac{\text{cis}\theta}{\text{cis}\phi} = \text{cis}(\theta - \phi)$

$$\therefore \frac{\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)^4}{\cos\frac{5\pi}{12} - \mathrm{i}\sin\frac{5\pi}{12}} = \frac{\left(\text{cis}\frac{\pi}{12}\right)}{\text{cis}\frac{5\pi}{12}} = \text{cis}\left(-\frac{\pi}{12}\right)$$

$= \cos\frac{\pi}{12} - \mathrm{i}\sin\frac{\pi}{12}$ *[5 marks]*

3. Let $z = r\mathrm{e}^{\mathrm{i}\theta}$,

$\Rightarrow\ z^3 = r^3\mathrm{e}^{3\mathrm{i}\theta}$

$|-4 + 4\sqrt{3}\,\mathrm{i}\,| = 8$

$\text{Arg}(-4 + 4\sqrt{3}\,\mathrm{i}\,) = \arctan(-\sqrt{3}\,) = -\frac{\pi}{3}$

$\therefore\ r^3\mathrm{e}^{3\mathrm{i}\theta} = 8\mathrm{e}^{-\mathrm{i}\pi/3}$

$$\Rightarrow\ r = 2,\ \theta = \frac{\left(-\frac{\pi}{3} + 2k\pi\right)}{3} = -\frac{\pi}{9}, \frac{5\pi}{9}, -\frac{7\pi}{9}$$

$\Rightarrow\ z = 2\mathrm{e}^{-7\mathrm{i}\pi/3}, 2\mathrm{e}^{-\mathrm{i}\pi/3}, 2\mathrm{e}^{5\mathrm{i}\pi/3}$ *[8 marks]*

4. (a) By De Moivre's theorem, $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$.

$\Rightarrow\ z^n - z^{-n} = \cos n\theta + \mathrm{i}\sin n\theta - (\cos n\theta - \mathrm{i}\sin n\theta) = 2\mathrm{i}\sin n\theta$

(b) $(z - z^{-1})^5 = z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$

$= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$

(c) But by (a), $(z - z^{-1})^5$ also equals $(2\mathrm{i}\sin\theta)^5 = 32\mathrm{i}\sin^5\theta$.

$\therefore\ 32\mathrm{i}\sin^5\theta = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$

But by (a), $(z^n - \mathrm{z}^{-n}) = 2\mathrm{i}\sin n\theta$.

$\therefore\ 32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$

$\Rightarrow\ \sin^5\theta = \dfrac{1}{16}\sin 5\theta - \dfrac{5}{16}\sin 3\theta + \dfrac{5}{8}\sin\theta$ *[12 marks]*